Appendix to Chapter 2

An infinite series can only be differentiated term-by-term if the resulting series converges uniformly. Thus the derivation of

$$-\frac{\zeta'}{\zeta}(\sigma) = \sum_p \sum_{r\geq 1} \frac{\log p}{p^{r\sigma}},$$

given in the notes for $\sigma > 1$ can only be justified by the following result.

Example 2.26 *The series*

$$\sum_p \frac{1}{\left(1-\frac{1}{p^\sigma}\right)} \frac{\log p}{p^\sigma}$$

converges ***uniformly*** *for $\sigma \geq 1+\delta$ for any $\delta > 0$.*

Proof

$$\frac{1}{p^\sigma} \leq \frac{1}{2^\sigma} \leq \frac{1}{2} \quad \text{and so} \quad \left(1-\frac{1}{p^\sigma}\right)^{-1} \leq 2.$$

Thus

$$\sum_p \frac{1}{\left(1-\frac{1}{p^\sigma}\right)} \frac{\log p}{p^\sigma} \leq 2\sum_p \frac{\log p}{p^\sigma} \leq 2\sum_{n=1}^{\infty} \frac{\log n}{n^\sigma}.$$

This resulting sum over all integers has been shown to converge *uniformly* for $\sigma \geq 1+\delta$ for any $\delta > 0$ in the Background: Complex Analysis II notes. We repeat it here: Let $M_n = (\log n)/n^{1+\delta}$. Then

$$\left|\frac{\log n}{n^s}\right| = \frac{\log n}{n^\sigma} \leq \frac{\log n}{n^{1+\delta}} = M_n.$$

By looking for a turning point for $(\log x)/x^{1+\delta}$ we know that $(\log n)/n^{1+\delta}$ is decreasing for $n \geq n_0$, where n_0 is the least integer greater than $\exp(1/(1+\delta))$. For such n

$$\frac{\log n}{n^{1+\delta}} \leq \int_{n-1}^{n} \frac{\log t}{t^{1+\delta}} dt.$$

Hence

$$\sum_{n\geq n_0} M_n \leq \int_{n_0-1}^{\infty} \frac{\log t}{t^{1+\delta}} dt,$$

a convergent integral since $\delta > 0$. (Integration by parts will show this.) Hence $\sum_{n\geq n_0} M_n$ and thus $\sum_{n\geq 1} M_n$ converge. The result then follows from the Weierstrass M-test. ■

Proofs of Lemma 2.18 and Corollary 2.21.

Lemma 2.18 *Chebyshev's inequality For all* $\varepsilon > 0$

$$(\log 2 - \varepsilon)\, x < \theta(x) < (2\log 2 + \varepsilon)\, x$$

for all $x > x_3\,(\varepsilon)$.

Proof Let $\varepsilon > 0$ be given. Lemma 2.17 means that there exists a function $\mathcal{E}(x) : \psi(x) = \theta(x) + \mathcal{E}(x)$ and $|\mathcal{E}(x)| < Cx^{1/2}$ for some constant $C > 0$. Yet $Cx^{1/2} \le \varepsilon x/2$ for x sufficiently large, i.e. $x > x_2\,(\varepsilon)$. Thus, for such x,

$$\psi(x) - \varepsilon x/2 \le \theta(x) \le \psi(x) + \varepsilon x/2.$$

Next apply Corollary 2.16 with $\varepsilon/2$ in place of ε, to get

$$(\log 2 - \varepsilon/2)\, x - \varepsilon x/2 \le \theta(x) \le (2\log 2 + \varepsilon/2)\, x + \varepsilon x/2,$$

valid for $x > \max\,(x_1\,(\varepsilon/2), x_2\,(\varepsilon))$. ■

Corollary 2.21 *Chebyshev's inequality For all* $\varepsilon > 0$

$$(\log 2 - \varepsilon)\, \frac{x}{\log x} < \pi(x) < (2\log 2 + \varepsilon)\, \frac{x}{\log x}$$

for all $x > x_4\,(\varepsilon)$.

Proof Let $\varepsilon > 0$ be given. Theorem 2.20 means that there exists a function $\mathcal{E}(x) : \pi(x) = \theta(x)/\log x + \mathcal{E}(x)$ where $|\mathcal{E}(x)| < Cx/\log^2 x$ for some constant $C > 0$. Yet $C/\log x \le \varepsilon/2$ for x sufficiently large, i.e. $x > x_5\,(\varepsilon)$. Thus, for such x,

$$\frac{\theta(x) - \varepsilon x/2}{\log x} \le \pi(x) \le \frac{\theta(x) + \varepsilon x/2}{\log x}.$$

Next apply Lemma 2.18 with $\varepsilon/2$ in place of ε, to get$(\log 2 - \varepsilon)\, x < \theta(x) < (2\log 2 + \varepsilon)\, x$

$$\frac{(\log 2 - \varepsilon/2)\, x - \varepsilon x/2}{\log x} \le \pi(x) \le \frac{(2\log 2 + \varepsilon/2)\, x + \varepsilon x/2}{\log x},$$

valid for $x > \max\,(x_3\,(\varepsilon/2), x_5\,(\varepsilon))$. ■

Inequalities between $\pi(x)$ and $\theta(x)$.

In Theorem 2.20 we gave an asymptotic relation between $\pi(x)$ and $\theta(x)$. We can, instead give a simple inequality,

$$\theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x.$$

What is not simple is a lower bound on θ in terms of π.

Lemma 2.27 *For all* $0 < \alpha < 1$

$$\pi(x) - \pi(x^\alpha) \leq \frac{1}{\log x^\alpha} \left(\theta(x) - \theta(x^\alpha)\right). \tag{25}$$

Proof Given $0 < \alpha < 1$, we have

$$\pi(x) - \pi(x^\alpha) = \sum_{x^\alpha < p \leq x} 1.$$

For the primes p counted in this sum we have $x^\alpha < p$ which can be rewritten as

$$1 < \frac{\log p}{\log x^\alpha}.$$

Thus

$$\begin{aligned} \sum_{x^\alpha < p \leq x} 1 \leq \sum_{x^\alpha < p \leq x} \frac{\log p}{\log x^\alpha} &= \frac{1}{\log x^\alpha} \sum_{x^\alpha < p \leq x} \log p \\ &= \frac{1}{\log x^\alpha} \left(\theta(x) - \theta(x^\alpha)\right). \end{aligned}$$

■

These inequalities can be used to deduce Chebyshev's inequality for π from Chebyshev's inequality for θ. So, start from the result that for all $\varepsilon > 0$

$$(\log 2 - \varepsilon)\, x < \theta(x) < (2 \log 2 + \varepsilon)\, x \tag{26}$$

for all $x > x_3(\varepsilon)$. Then from $\theta(x) \leq \pi(x) \log x$ we get the lower bound on $\pi(x)$:

$$(\log 2 - \varepsilon) \frac{x}{\log x} < \pi(x)$$

for all $x > x_3(\varepsilon)$.

For the upper bound we start from (25) with α to be chosen. Simplify slightly, so

$$\pi(x) \leq \pi(x^\alpha) + \frac{1}{\log x^\alpha}\theta(x)\,.$$

Then use the trivial $\pi(x) \leq x$ along with the upper bound in (26), though with ε replace by $\varepsilon/2$, so

$$\pi(x) \leq x^\alpha + \frac{(2\log 2 + \varepsilon/2)\,x}{\alpha \log x}.$$

for $x > x_3\,(\varepsilon/2)$. Now choose $\alpha < 1$ sufficiently close to 1 that

$$\frac{(2\log 2 + \varepsilon/2)}{\alpha} = 2\log 2 + \frac{3\varepsilon}{4},$$

i.e.

$$\alpha = \frac{2\log 2 + \varepsilon/2}{2\log 2 + 3\varepsilon/4} = 1 - \frac{\varepsilon}{8\log 2 + 3\varepsilon}.$$

Then for such α we have

$$\pi(x) \leq x^\alpha + \left(2\log 2 + \frac{3\varepsilon}{4}\right)\frac{x}{\log x}.$$

Our choice of α is still < 1 so

$$x^\alpha \leq \frac{\varepsilon}{4}\frac{x}{\log x},$$

for x sufficiently large, i.e. $x > x_6\,(\varepsilon)$. Combining we find that

$$\pi(x) \leq (2\log 2 + \varepsilon)\,\frac{x}{\log x}$$

for $x > \max\left(x_3\,(\varepsilon/2), x_6\,(\varepsilon)\right)$.